Suggested Solution to Quiz 2

April 7, 2016

1. (**5 points**) Can the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), & 0 < x < 1 \\ X'(0) = 0, \quad X(1) = 0 \end{cases}$$

have nonpositive eigenvalues? Prove your statements. Write down all the eigenvalues and corresponding eigenfunctions.

Solution:

(a) No, the above eigenvalue problem only have positive eigenvalues.
In fact, let λ be the eigenvalue of the problem and $X(x)$ the corresponding eigenfunction. Multiply the equation $-X''(x) = \lambda X(x)$ by $X(x)$ and integrate with respect to x, then we get

$$-\int_0^1 X''(x)X(x)dx = \lambda \int_0^1 X^2(x)dx$$

With the help of the boundary conditions, we have

$$-\int_0^1 X''(x)X(x)dx = -X'(x)X(x)\Big|_0^1 + \int_0^1 |X'(x)|^2 dx = \int_0^1 |X'(x)|^2 dx$$

Therefore,

$$\lambda = \frac{\int_0^1 |X'(x)|^2 dx}{\int_0^1 X^2(x)dx} \geq 0$$

If $\lambda = 0$, then we must have $X'(x) \equiv 0$ on $[0,1]$ which implies that $X(x) = Constant$. Since $X(1) = 0$, then $X(x) \equiv 0$ which is impossible. Therefore $\lambda > 0$.

(b) Since $\lambda > 0$, let $\lambda = \beta^2, \beta > 0$. Then the general solutions of $-X''(x) = \lambda X(x)$ are

$$X(x) = A\cos\beta x + B\sin\beta x$$

Hence

$$0 = X(1) = A\cos\beta + B\sin\beta$$
$$0 = X'(0) = B\beta$$

Thus $B = 0$ and $\cos\beta = 0$. Therefore the eigenvalues are $\lambda_n = (\frac{\pi}{2} + n\pi)^2$ and corresponding eigenfunctions are $X_n(x) = \cos(\frac{\pi}{2} + n\pi)x$ for $n = 1, 2, \ldots$.

2. (**5 points**) Find the Fourier cosine series of $f(x) = x$ on $(0, \pi)$. Then find the sum

$$\sum_{k=0}^{\infty} (\frac{1}{2k+1})^4 = 1 + (\frac{1}{3})^4 + (\frac{1}{5})^4 + (\frac{1}{7})^4 + \cdots$$

by using Parseval's equality.

Solution: The Fourier cosine series of $f(x) = x$ on $(0, \pi)$ is

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

where the coefficients are

$$A_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$$

and

$$\begin{aligned} A_n =& \frac{2}{\pi}\int_0^{\pi} x \cos nx dx \\ =& \frac{2}{n\pi} x \sin nx \Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nx dx \\ =& \frac{2}{n^2\pi}[(-1)^n - 1], \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \frac{\pi}{2} - \frac{4}{\pi}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots).$$

The Parseval's equality is

$$\int_0^{\pi} |f(x)|^2 dx = \sum_{n=0}^{\infty} |A_n|^2 \int_0^{\pi} |X_n(x)|^2 dx$$

Here $f(x) = x$ and $X_0(x) = \frac{1}{2}, X_n(x) = \cos nx, n = 1, 2, \cdots$, thus we have

$$\frac{\pi^3}{3} = \pi^2\frac{\pi}{4} + \frac{\pi}{2}\frac{16}{\pi^2}[1 + (\frac{1}{3})^4 + (\frac{1}{5})^4 + (\frac{1}{7})^4 + \cdots]$$

Hence

$$\sum_{k=0}^{\infty}(\frac{1}{2k+1})^4 = \frac{\pi^4}{96}.$$